MT2504 Revision

General

- Be relatively familiar with where things are in the notes.
  So when it comes to a question you know which section the question relates to and can look at that section for methods/ideas
- Do the past papers
  Do the problem sheets (even the extra qs)
  Redo the class test
  Try + find common methods

- MMS - D2017, D2018, A2018, A2019
  - solns + pp
  online - D2014, D2015, A2015
  - no solns
- This session -
  * Double counting
  * Bijections
  * Derangements
  * Qs from 2014/2015

  } inc "story" proofs

Double counting

- General idea;
  * count 1 way, then count another
  * sum not ⇒ there is a variable!

Example 1 - Show that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Consider $X=\{1,2,\dots,n\}$, $S\subseteq X$ of size k, $x\in S$. Count (x,S)

Soln

count 1 — choose S then x

S has size $k \Rightarrow \binom{n}{k}$ ways to choose S

$x\in S \Rightarrow \binom{k}{1}=k$ ways to choose $x\in S$

$\Rightarrow k\binom{n}{k}$ ways to choose (x,S)

count 2 — choose x then S

There are $\binom{n}{1}=n$ ways to choose x

Since $x\in S$ we now want to choose $S\setminus\{x\}$ from $\{1,2,\dots,n\}\setminus\{x\}$

$\Rightarrow k-1$ from $n-1$

$\Rightarrow \binom{n-1}{k-1}$

$\Rightarrow n\binom{n-1}{k-1}$ ways to choose (x,S)

$\Rightarrow k\binom{n}{k} = n\binom{n-1}{k-1}$ $\square$

Example 2 – Vandemode's identity – Show that

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Consider m maths students and n physics students, form a class of size r

Soln

- Count 1 – total population $= m+n$, choosing a class of size r $\Big\} \Rightarrow \binom{m+n}{r}$
- Count 2 – Sum $\Rightarrow$ need a variable!

  Let k = number of maths students in class

  $\Rightarrow r-k$ = number of physics students

  $\Rightarrow \binom{m}{k}\binom{n}{r-k}$ ways to choose the class if k maths students

  k can be any value between 0 and r

  $\Rightarrow$ total options $= \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$ $\square$

Example 3 – Show that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$$

Consider (X,Y) disjoint subsets of $\{1,2,\ldots,n\}$ with $|X \cup Y| = m$

Soln

- Count 1 – choose $X \cup Y$ then choose X

  $X \cup Y$ has size m $\Rightarrow \binom{n}{m}$ ways to choose $X \cup Y$

  how many ways to choose X?

  $X \subseteq X \cup Y$, so we're really asking how many subsets of $X \cup Y$?

  Each el of $X \cup Y$ can either be in or out of X

  $\Rightarrow$ 2 options for each el

  $\Rightarrow 2^m$ ways to choose X

  $\Rightarrow 2^m\binom{n}{m}$ ways to choose (X,Y)

- Count 2 – Sum $\Rightarrow$ need a variable

  Let $|X| = k \Rightarrow |Y| = m-k$ since X, Y disjoint

  n points to choose X from $\Rightarrow \binom{n}{k}$ choices for X

  $n-k$ points to choose Y from $\Rightarrow \binom{n-k}{m-k}$ choices for Y

  $\Rightarrow \binom{n}{k}\binom{n-k}{m-k}$ choices for (X,Y) when $|X| = k$

  $0 \le k \le m$

  $\Rightarrow \sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k}$

but question wants $\sum_{k=0}^{n}$ eek!!

If $k+1 \le m \le n$ then $\binom{n-k}{m-k} = 0$

$$\Rightarrow \sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{m-k} \qquad \square$$

Bijection Questions

- A bij is injective (1-1) and surjective (onto)
- A func is a bij $\Leftrightarrow$ it's invertible
- $\exists\, f: X \to Y$ a bij $\Rightarrow |X| = |Y|$

Example 1 - Show that

$$|\{(a,b,c) \mid a,b,c \in \mathbb{N},\ 1 \le a < b < c \le n\}| = \binom{n}{3}$$

Soln

Let X = all subsets of size 3 from $\{1,2,\ldots,n\}$

$Y = \{(a,b,c) \mid a,b,c \in \mathbb{N},\ 1 \le a < b < c \le n\}$

We know that $|X| = \binom{n}{3}$

$\Rightarrow$ want to find a bij between X and Y

Let $f: X \to Y$

$f(\{x_1,x_2,x_3\}) = (a,b,c)$ where $a = \min(\{x_1,x_2,x_3\})$
$c = \max(\{x_1,x_2,x_3\})$
$b = \{x_1,x_2,x_3\} \setminus \{a,b\}$

since $|\{x_1,x_2,x_3\}| = 3$

$\Rightarrow x_1, x_2, x_3$ distinct

$\Rightarrow a,b,c$ well defined

$\Rightarrow a < b < c$

Since $x_1, x_2, x_3 \in \{1 \ldots n\}$

$\Rightarrow 1 \le a < b < c \le n$

$f^{-1}: Y \to X$

$f^{-1}((a,b,c)) = \{a,b,c\}$

Since $a < b < c \Rightarrow |\{a,b,c\}| = 3$

since $1 \le a,b,c \le n \Rightarrow \{a,b,c\} \subseteq \{1 \ldots n\}$

Clearly $f^{-1} \circ f = 1 = f \circ f^{-1}$

$\Rightarrow f$ a bijection

$\Rightarrow \quad |X| = |Y|$

$\Rightarrow \quad |Y| = \binom{n}{3} \qquad \square$

Example 2 - Show that
$|\{(a,b,c) \mid a,b,c \in \mathbb{N} \quad 1 \leq a \leq b \leq c \leq n\}| = \binom{n+2}{3}$

Soln

Let X be the number of subsets of $\{1, 2, \dots n+2\}$ of size 3
$\Rightarrow |X| = \binom{n+2}{3}$

Let $Y = |\{(x, y, z) \mid x,y,z \in \mathbb{N} \quad 1 \leq x < y < z \leq n+2\}|$
by Example 2 with $n+2$ in place of n
$\Rightarrow |X| = |Y|$

Let $Z = \{(a,b,c) \mid a,b,c \in \mathbb{N} \quad 1 \leq a \leq b \leq c \leq n\}$

Lets try and show $|Y| = |Z|$
Let $g : Z \rightarrow Y$

$g((a,b,c)) = (a, b+1, c+2)$

$1 \leq a \leq b \leq c \leq n \Rightarrow 1 \leq a < b+1 < c+2 \leq n$
$\Rightarrow g((a,b,c)) \in Y$

$g^{-1}((x,y,z)) = (x, y-1, z-2)$

$1 \leq x < y < z \leq n+2 \Rightarrow 1 \leq x \leq y-1 \leq z-2 \leq n$
$\Rightarrow g^{-1}((x,y,z)) \in Z$

Check that $g \circ g^{-1} = 1 = g^{-1} \circ g$

$\Rightarrow g$ a bij
$\Rightarrow |Z| = |Y|$
$\Rightarrow |Z| = \binom{n+2}{3} \qquad \square$

Past Exam Questions

- These are 2014/2015 so the course might have changed a lot since then so take all these papers with a pinch of salt

Example 1

3. A car park attendant loses the name tags for a set of seven car keys, so hands out the keys randomly when the seven owners return.

(a) How many ways are there of returning the keys? **[1]**

(b) How many ways are there of returning the keys so that nobody receives their own keys? **[2]**

(c) How many ways are there of returning the keys so that at least two people receive their own keys? **[2]**

a) 1^{st} person → 7 choices for keys
2^{nd} person → 6 choices for keys (can't have the same as p1)
⋮ ⋮
6^{th} person → 2 choices (can't have the same as p1, p2, ..., p5)
7^{th} person → 1 choice (only 1 key left)
⇒ $7 \times 6 \times \dots \times 2 \times 1 = 7!$ choices

b) Let X be a collection of k people
Assume we want these k people to get the right key
⇒ for these k people only 1 option of key
7-k people left and 7-k keys left to give out
⇒ $(7-k)!$ ways to give out these remaining keys
⇒ by inclexc # of ways to give at least 1 person the right key is

$$\sum_{k=1}^{7} (-1)^{k-1} \binom{7}{k} (7-k)!$$

⇒ The number of ways for no one to have the right key is

$$7! - \sum_{k=1}^{7} (-1)^{k-1} \binom{7}{k} (7-k)!$$

$$= (-1)^0 \binom{7}{0} (7-0)! + \sum_{k=1}^{7} (-1)^{k} \binom{7}{k} (7-k)!$$

$$= \sum_{k=0}^{7} (-1)^{k} \binom{7}{k} (7-k)!$$

(agrees with thm 6.2 in notes)

c) At least 2 people ✓ = total − no one gets ✓ − exactly 1 person ✓

total = 7! by (a)

no one happy $= \sum_{k=0}^{7} (-1)^k \binom{7}{k} (7-k)!$

exactly 1 $= \binom{7}{1} \cdot \sum_{k=0}^{6} (-1)^k \binom{6}{k} (6-k)!$

ways to choose that person

derangements on the remaining 6 people

$\Rightarrow$ At least 2 people get the right key $= 7! - \sum_{k=0}^{7} (-1)^k \binom{7}{k} (7-k)! - \binom{7}{1} \sum_{k=0}^{6} (-1)^k \binom{6}{k} (6-k)!$

(Similar to ex 6.3)

Example 2

2. This question concerns a standard 52-card pack of cards.

(a) What is the probability that a 7-card hand contains exactly one ace? [2]

(b) How many 7-card hands contain four cards with the same value (e.g. four threes)? [1]

(c) How many 7-card hands contain four cards of one suit, and three cards of a different suit? [2]

(d) How big does a hand need to be to guarantee that it will contain at least three cards of the same suit? [1]

a) 4 aces $\Rightarrow$ $52 - 4 = 48$ non-aces

hands with 1 ace $= \binom{4}{1} \binom{48}{6}$

choice of ace

choice of remaining cards

total hands $= \binom{52}{7}$

$\Rightarrow P(1 \text{ ace exactly}) = \dfrac{\binom{4}{1}\binom{48}{6}}{\binom{52}{7}}$

b) 13 values

\# choices for cards of the same value $\binom{13}{1}$

\# choices for remaining cards $\binom{52-4}{3} = \binom{48}{3}$

$\Rightarrow \binom{13}{1}\binom{48}{3}$

(there are $\binom{4}{4} = 1$ ways to choose the first 4 cards once we know the value)

c) 4 suits, 13 cards of 1 suit

$\Rightarrow \binom{13}{4} \cdot \binom{52-13}{3} = \binom{13}{4} \cdot \binom{39}{3}$

4 cards of the same suit

3 cards of a diff suit

d) "worst" case you keep choosing cards with diff suits

eg after 8 you have exactly 2 of each suit

The next card must make 3 of one suit
$\Rightarrow$ need at least 9 cards!

Example 3

3. Consider the alphabet $A = \{0, 1\}$.

(a) How many words over A are there of length 5? **[1]**

(b) How many words over A are there of length 10 with exactly 5 0s and 5 1s? **[1]**

(c) Prove that the number of words over A containing exactly a 0s and b 1s is equal to the number of solutions to the equation $y_1 + y_2 + \cdots + y_{b+1} = a$ in non-negative integers. **[3]**

a) word length 5
each letter 0 or 1
$\Rightarrow$ 2 choices for each letter
$\Rightarrow 2^5$ words length 5 over a

b) words length 10 with 5 0s + 5 1s

Image $\underline{\ \ }_{l_1}\ \underline{\ \ }_{l_2}\ \underline{\ \ }_{l_3}\ \underline{\ \ }_{l_4}\ \underline{\ \ }_{l_5}\ \underline{\ \ }_{l_6}\ \underline{\ \ }_{l_7}\ \underline{\ \ }_{l_8}\ \underline{\ \ }_{l_9}\ \underline{\ \ }_{l_{10}}$

10 slots, one for each letter l_i
once we choose where to put the 0s
it follows that the 1s just have to go in the remaining slots
$\Rightarrow$ same as ways to choose a set of size 5 from $\{l_1, l_2, \ldots l_{10}\}$
$\Rightarrow \binom{10}{5}$

c) word cont a 0s and b 1s
$\Rightarrow$ word has length $a+b$

by the arg above the number of words is;

$$\binom{a+b}{a} = \binom{a+b}{b}$$

consider solns to

$$y_1 + y_2 + \dots + y_{b+1} = a$$

st $y_i \in \mathbb{N}$ $y_i \geq 0$

(see thm 7.6 in the notes but we'll still cover it here ☺)

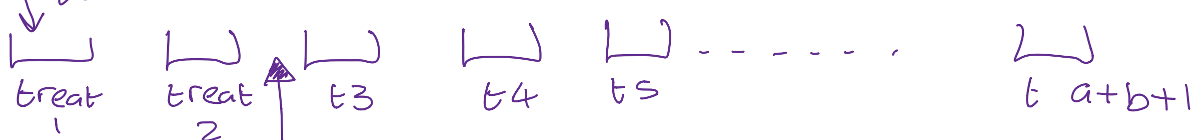
Assume we have b+1 dogs and a+b+1 treats

we're not monsters so we want to make sure that each puppy gets at least one treat

Assume that each puppy gets all its treats in 1 go

consider we give the treat to which puppy

this is a spacer to show



at each gap we can choose to stay with the current puppy or move on.

Since each puppy gets at least 1 treat and we cannot go back to previous puppies we must move on exactly b times (b+1 puppies)

There are $a+b$ choices where we can move (#gaps = # trees -1)

$\Rightarrow \binom{a+b}{b}$ choices for when to move

$\Rightarrow \binom{a+b}{b}$ ways to give the $a+b+1$ treats to $b+1$ dogs

Let x_i = # treats dog i gets

Since each dog gets at least 1 treat $x_i \geq 1$

Let $y_i = x_i - 1$

Since $x_i \geq 1 \Rightarrow y_i \geq 0$

Since

$$x_1 + x_2 + \dots + x_{b+1} = a+b+1$$

$$\Rightarrow (y_1+1) + (y_2+1) + \dots + (y_{b+1}+1) = a+b+1$$

$$\Rightarrow y_1 + y_2 + \dots + y_{b+1} + b+1 = a+b+1$$

$$\Rightarrow y_1 + y_2 + \dots + y_{b+1} = a$$

$\Rightarrow$ bij between our puppy prob and part c

$\Rightarrow$ ans $= \binom{a+b}{b}$

= number of words with a 0s and b 1s

Good Luck with all your exams!

This is meant to be a 4 leaf clover!